TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 302, Number 1, July 1987

MARTIN BOUNDARIES OF RANDOM WALKS: ENDS OF TREES AND GROUPS

MASSIMO A. PICARDELLO AND WOLFGANG WOESS

ABSTRACT. Consider a transient random walk X_n on an infinite tree T whose nonzero transition probabilities are bounded below. Suppose that X_n is uniformly irreducible and has bounded step-length. (Alternatively, X_n can be regarded as a random walk on a graph whose metric is equivalent to the metric of T.) The Martin boundary of X_n is shown to coincide with the space Ω of all ends of T (or, equivalently, of the graph). This yields a boundary representation theorem on Ω for all positive eigenfunctions of the transition operator, and a nontangential Fatou theorem which describes their boundary behavior. These results apply, in particular, to finitely supported random walks on groups whose Cayley graphs admit a uniformly spanning tree. A class of groups of this type is constructed.

1. Introduction. Let X_n, $n = 0, 1, 2, \dots$, be a discrete-time Markov chain, here usually called "random walk", with values in a countable state space T. Suppose X_n is transient. Fix a reference state e in T, choose u, v in T, denote by $G(u,v)$ the expected number of visits to v starting from u (*Green function*), and by $K(u,v) = G(u,v)/G(e,v)$ the *Martin kernel*. The *Martin compactification* $\hat{T}$ of T with respect to (X_n) is the smallest compactification of T where all the functions $K(u,\cdot)$ extend continuously; the set of new points $T^* = \hat{T} - T$ is the *Martin boundary.* Every positive function on T which is harmonic for the Markov transition operator can be represented as an integral of the Martin kernel with respect to some Borel measure on T^*. The *Poisson boundary* is the support of the measure on T^* representing the harmonic function with constant value one.

The reader is referred to [**KSK**, **Re**] for more details, and to [**AC**] for a bibliography on the various concepts of boundaries of Markov chains. A stimulating overview of group-invariant boundary theory and additional references can be found in [**KV**].

The aim of the present paper is to give, under suitable assumptions, a geometrical description of the Martin boundary T^* as the space Ω of all "ends" of a tree whose vertices constitute the state space.

An extensive investigation of the Martin boundary of "nearest neighbor" random walks on trees is due to P. Cartier [**Ca**]. If T is a homogeneous tree of even degree q, then it is the Cayley graph of the free group $\mathbf{F}$ with $q/2$ generators. In this setting, particular attention has been devoted to Markov chains on T whose transition

Received by the editors August 12, 1985 and, in revised form, June 17, 1986.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 60J50; Secondary 05C05, 20F32, 43A05, 60J15.

Key words and phrases. Random walk on a tree, Martin boundary, harmonic functions, ends of a tree, ends of a group.

The second author was supported by Consiglio delle Ricerche, GNAFA.

probabilities are invariant under **F**, that is, to group-invariant random walks on **F**. For nearest neighbor random walks of this type, the Martin boundary was studied by E. Dynkin and M. Malyutov [**DM**], and the special case of isotropic random walks and their "hitting distribution" on the boundary was considered by H. Furstenberg [**Fu**]. Finally, Y. Derriennic [**De**] studied in full detail all finitely supported **F**-invariant random walks by an approach which extends the method of [**Ca**] in the setting of a free group. It turns out in [**De**] that, for **F**-invariant random walks, the Martin boundary can be realized as the space of all "ends" of the tree, or free group respectively. In a similar way, C. Series [**Ser**] determined the Martin boundaries for finitely generated Fuchsian groups as a subset of the unit circle.

The present paper, extending the results of [**Ca**, **De**], considers random walks of finite range on general trees which are not necessarily homogeneous. Three assumptions are needed for our approach: that X_n is "uniformly irreducible", has "bounded step-length" (see §2) and that the nonzero one-step transition probabilities are bounded away from zero. These conditions hold automatically for finitely supported group-invariant random walks.

We give examples which show that the first two conditions are necessary for our approach (§6, Examples 1 and 2). The third is not necessary for nearest neighbor random walks on trees [**Ca**]: in the general case, we need it to counteract the fact that the Martin kernel is not locally constant.

The tools which we need are developed in §§2 and 3. We consider "intermediate sets" (that is, finite sets that disconnect the graph of the Markov chain in a uniform way) and corresponding "intermediate matrices", as introduced in [**De**]. Using our three assumptions, we prove a modified version of the Perron-Frobenius theorem [**Bi**, **Sen**], concerning projective convergence of infinite products of intermediate matrices. Our result applies to matrices which are not necessarily square, and deals with a possibly infinite collection of different intermediate matrices (§3, Theorem 1).

On the basis of this result, we show that *the Martin boundary can be realized as the set Ω of all ends of the tree* (§4, Theorem 2). However, contrary to the group-invariant cases of [**De**, **Ser**] and of §5, the Poisson boundary may in general be properly contained in Ω (§6, Example 3). We also show that all points of Ω are extreme: This yields uniqueness of the integral representation of positive harmonic functions (§4, Proposition 4). We prove a "nontangential Fatou theorem" for convergence to the boundary of harmonic functions on T (§4, Theorem 3), which extends results of [**Ca**, **De**, **KP**]. These results also apply to all positive eigenfunctions of the transition operator (§4, Corollary 7).

In §5, we turn our attention to group-invariant random walks. Contrary to the suggestion in [**DM**], the realization of the Martin boundary cannot be easily transported from free groups to all finitely generated discrete groups. Rather than restricting attention to specific examples, we consider the class of all countable, finitely generated groups whose Cayley graphs admit a "uniformly spanning tree" (§5, Definition 2). Every finitely supported, irreducible random walk on a group in this class, considered as a walk on the tree, satisfies our three conditions (§5, Lemma 8). In particular, the Martin boundary is the space of all ends of the tree, or, equivalently, of all ends of the group [**Fr**, **St**], in accordance with an observation of [**K3**]. All results of §§3–4 carry over to these groups. We show that the class of groups which have a uniformly spanning tree is closed with respect to amalgamations and HNN-extensions over finite subgroups (§5, Theorem 4).

Inspiration for the present paper came from Y. Derriennic's paper [**De**]. The methods of [**De**] rely crucially on the fact that, due to the nonamenability of the free group, the Green function is square summable. In general, we do not have this tool at our disposal, and it turns out that all the relevant results rely only on the underlying tree-structure. In this paper, topics in probability and analysis are studied in a spirit which is more typical of combinatorial graph theory. We acknowledge the influence of W. Imrich and N. Seifter on this way of thinking.

Finally, we acknowledge helpful suggestions of the referee concerning the exposition.

2. Preliminaries. Throughout this paper, unexplained graph-theoretical terminology is as in [**Ca**]. Let T be a locally finite, infinite tree. Given two vertices u and v, there is a unique (finite) *geodesic* $[u = u_0, u_1, \ldots, u_k = v]$ of successively contiguous vertices u_i in T connecting u and v. The *distance* between u and v, denoted by $d(u,v)$, is then the number k of edges on the geodesic.

We consider a transition operator P, acting on real functions defined on the vertices of T by the rule $Pf(u) = \sum_{v \in T} p(u,v) f(v)$, where $(p(u,v))_{u,v\in T}$ is a nonnegative stochastic matrix: $\sum_v p(u,v) = 1$ for every u. The matrix associated with the nth iterate P^n of P is denoted by $(p^{(n)}(u,v))_{u,v\in T}$. We assume that the transition operator is related to the tree structure by the following three basic properties:

(i) Uniform irreducibility: there is an integer M such that, for any pair u, v of neighbors, there is a $k \le M$ (depending on u and v) such that $p^{(k)}(u,v) > 0$.

(ii) Bounded step-length: there is an integer s such that $p(u,v) > 0$ implies $d(u,v) \le s$.

(iii) There is a bound $\delta > 0$ such that $p(u,v) > 0$ implies $p(u,v) \ge \delta$.

Associated with P there is the usual product measure space $(\Lambda, \mathcal{F})$ and the family $\Pr_u, u \in T$, of probability measures which govern a Markov chain X_n, $n = 0, 1, 2, \ldots$, of T-valued random variables defined on $(\Lambda, \mathcal{F})$ according to the rule

$$\Pr_u[X_n = v] = p^{(n)}(u,v);$$

the index u of $\Pr_u$ refers to the condition $X_0 = u$. Thus, X_n can be considered as the random vertex visited at time n, and we shall speak of a *random walk* on T.

The probabilistic significance of condition (i) is the following: after starting at any vertex (= state) u, each vertex v of T can be reached with positive probability in a finite number of steps which is bounded by a constant multiple of $d(u,v)$. This corresponds to the notation of irreducibility in the group-invariant case [**De**, **Ser**]. In particular, all states communicate. Note that, by the homogeneity condition (iii), $p^{(k)}(u,v) > 0$ implies $p^{(k)}(u,v) \ge \delta^k$.

Let us now consider the average $\overline{P}$ of the first M powers of P (M as in (i)). Then, if u and v are neighbors, $\bar{p}(u,v) > 0$ by (i) and $\bar{p}(u,v) \ge \delta^M/M$ by (iii). Thus T has the following property:

LEMMA 1. *The degree of vertices of T (the number of neighbors) is uniformly bounded by M/δ^M.*

A *harmonic function* is a function h on T such that

$$h(u) = \sum_{v \in T} p(u,v)h(v) \tag{2.1}$$

for every u in T. We shall say that h is harmonic at u if (2.1) holds for the vertex u.

It is obvious that by (i), a nonnegative harmonic function that vanishes at a vertex must be identically zero. Moreover, observe that by (ii) and (iii), $\{v|p(u,v) > 0\}$ is a finite set for every u. Therefore, if we choose and fix, once for all, a reference vertex e in T, then the set $\mathcal{B}$ of positive harmonic functions h such that $h(e) = 1$ is compact in the topology of pointwise convergence: that is, the convex cone of positive harmonic functions has compact base.

LEMMA 2 (LOCAL HARNACK INEQUALITY). *Let h be a nonnegative function on T, harmonic outside $W \subset T$. If $d(u,W) > (M-1)s$ and v is a neighbor of u, then $h(u) \geq \delta^M h(v)$.*

PROOF. Observe that $d(u,u') \leq sj$ if $p^{(j)}(u,u') > 0$. In particular, if $j < M$ then $u' \notin W$. Therefore h is harmonic at u', and, inductively,

$$h(u) = \sum_{u' \in T} p^{(j)}(u,u')h(u') = \sum_{v' \in T} p^{(j+1)}(u,v')h(v').$$

By (i), there exists $k = j+1 \leq M$ such that $p^{(k)}(u,v) > 0$. Thus $h(u) \geq p^{(k)}(u,v)h(v) \geq \delta^k h(v)$. □

The *Green function* is

$$G(u,v) = \sum_{n=0}^{\infty} p^{(n)}(u,v), \qquad u,v \in T. \tag{2.2}$$

Starting at u, $G(u,v)$ is the expected number of visits to the vertex v during the lifetime of the random walk. Throughout §§2, 3, and 4 we assume that the random walk is *transient*, i.e. $G(u,v) < \infty$ for all u,v in T. Note that, by (i), $G(u,v) > 0$ for all u,v in T and, in fact, $G(u,v) \geq \delta^{Md(u,v)}$. This allows us to define the *Martin kernel* with respect to the reference vertex e:

$$K(u,v) = G(u,v)/G(e,v), \qquad u,v \in T. \tag{2.3}$$

Let

$$F(u,v) = \Pr[\exists n \geq 0 : X_n = v | X_0 = u], \qquad u,v \in T. \tag{2.4}$$

Then an easy application of the strong Markov property yields

$$F(u,u) = 1, \quad G(u,v) = F(u,v)G(v,v) \quad \text{and} \quad F(u,w)F(w,v) \leq F(u,v). \tag{2.5}$$

(See also [**Ca**, §2.4]. Note that the definition of $F(u,v)$ is not exactly the same as in [**Ca**, **Sen**, **GW**], where n is strictly positive in (2.4).) From (2.5) one obtains

$$K(u,v) = F(u,v)/F(e,v) \leq 1/F(e,u). \tag{2.6}$$

Following [**KSK**], we sketch the construction of the Martin boundary; details can be found in that reference. See also [**Re**].

By choosing a family of positive weights $q(u)$, $u \in T$ such that $\sum_{u \in T} q(u)/F(e,u) < \infty$, we can define a *Martin metric* d_M on T by

$$d_M(v,w) = \sum_{u \in T} q(u)|K(u,v) - K(u,w)|.$$

The topology induced by d_M does not depend on the particular choice of the weights $q(u)$ (see [**KSK**]):

(2.7) *(v_n) is a Cauchy sequence in (T, d_M) if and only if $(K(u,v_n))$ is a Cauchy sequence for every u in T.*

The *Martin* (exit) *boundary* of the random walk is the set $T^* = \hat{T} \setminus T$ where $\hat{T}$ is the completion of T with respect to the metric d_M. By continuity, the Martin kernel extends to a function on $T \times \hat{T}$, also denoted by $K(u,x)$.

For fixed $\tau \in T^*$, $K(\cdot, \tau)$ defines a harmonic function by property (iii), and the Poisson-Martin representation theorem says that for each *positive* harmonic function h there exists a Borel measure ν_h on T^* such that

$$h(u) = \int_{T^*} K(u,\tau)\nu_h(d\tau). \tag{2.8}$$

This integral representation can also be regarded as a consequence of Choquet's theory of convex cones. Indeed, (2.8) says that every extreme point of the convex base $\mathcal{B}$ defined above is of the type $K(\cdot,\tau)$ with $\tau \in T$. If in (2.8) we consider only measures supported on the set of all points τ such that $K(\cdot,\tau)$ is extreme in $\mathcal{B}$, then ν_h is unique. Finally, there exists a random variable X_∞ taking values in T^*, such that, in the topology of T,

$$\lim_{n \to \infty} X_n = X_\infty \quad \text{almost surely.} \tag{2.9}$$

We intend to provide a geometric realization of the Martin boundary. An *infinite geodesic* is a sequence $[v_0, v_1, v_2, \dots]$ of successively contiguous vertices without repetitions. Two infinite geodesics which have all but finitely many vertices in common are said to be *equivalent.* An equivalence class of infinite geodesics is called an *end.* The set of all ends of T is denoted by Ω. For every u in T and ω in Ω, there is exactly one representative $[v_0, v_1, v_2, \dots]$ of ω such that $v_0 = u$; it will be called the geodesic from u to ω. If $u \in T$ and x, y are different points of $T \cup \Omega$, then the *junction* $c(u,x,y)$ is the last common vertex of the geodesics from u to x and to y, respectively. A *branch* of T is a set $B(u,v) = \{x \in T \cup \Omega | v$ belongs to the geodesic from u to $x\}$, for different vertices u, v. The family of all branches of T is a subbasis of a topology making $T \cup \Omega$ a compact metrizable space that contains T as an open, dense subspace with the discrete topology. See [**Ca**, Chapter 1] for the details.

We shall show that Ω, with the induced topology, is in fact the Martin boundary of our random walk.

3. Intermediate sets and intermediate matrices. For every finite subset W of T, denote by S_W its first hitting time by the random walk. We define a substochastic transition operator Q_W by

$$Q_W f(v) = \sum_{w \in W} \Pr_v[X_{S_W} = w] f(w).$$

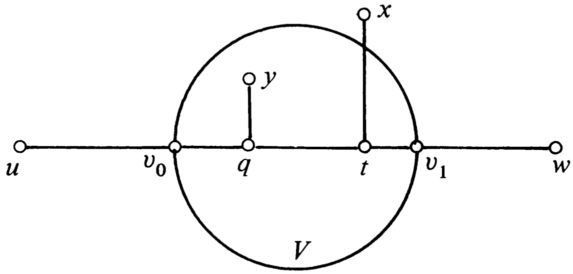

FIGURE 1

LEMMA 3. *If $d(u, W) > s$ then, for every function f on T, $Q_W f$ is harmonic at u.*

PROOF. If $p(u,v) > 0$ then $v \notin W$ by (ii). Now the strong Markov property yields $\Pr_u[X_{S_W} = w] = \sum_v p(u,v) \Pr_v[X_{S_W} = w]$, and the result follows. □

If V and W are finite subsets of T, then the truncation of Q_W onto V is associated with a finite-dimensional matrix

$$A(V, W) = (\alpha^W(v, w))_{v \in V, w \in W},$$

where $\alpha^W(v,w) = Q_W \delta_w(v)$. In particular, $\mathbf{f}(W, u)$ denotes the column vector $A(W, \{u\}) = (F(w,u))_{w \in W}$.

DEFINITION 1. (a) Let $v_0, v_1 \in T$, $d(v_0, v_1) = s - 1$. The set $V = V(v_0, v_1)$ of all vertices v with $d(v, v_0) \le s-1$ and $d(v, v_1) \le s - 1$ is called an *intermediate set.* It is said to lie between two points $u \in T$ and $x \in T \cup \Omega$ if v_0 and v_1 belong to the geodesic connecting u and x and do not coincide with the endpoints.

(b) An *intermediate matrix* is a matrix $A = A(V, W)$, where V and W are intermediate sets such that $d(V, W) > Ms$.

As T has bounded degree by Lemma 1, there is a uniform bound $M' \in \mathbf{N}$ for the number of elements of any intermediate set:

$$|V(v_0, v_1)| \le M'. \tag{3.1}$$

The following lemma is due to Derriennic [**De**].

LEMMA 4. *Let u, w be two vertices with $d(u,w) > s$, and let $V = V(v_0, v_1)$ be an intermediate set lying between u and w (see Figure 1). Then*

$$\Pr_u[X_n = w \text{ for some } n] = \Pr_u[X_n = w \text{ for some } n \text{ and } S_V < \infty].$$

In other words, the random walk starting from u must pass through V in order to visit w.

PROOF. We may assume that $d(u, v_0) < d(u, v_1)$ (see Figure 1). To begin with, one may exclude the trajectories $\lambda \in \Lambda$ which do not satisfy $d(X_j(\lambda), X_{j+1}(\lambda)) \le s$ for all $j \ge 0$. Now let S_0 be the first time $k \ge 0$ that the geodesic from u to X_k contains v_0 and $d(v_0, X_k) \ge d(v_1, X_k)$. Observe that $S_0 < \infty$ if the random walk hits w. We write $x = X_{S_0}$. We may assume that $d(v_0, x) \ge s$, as otherwise

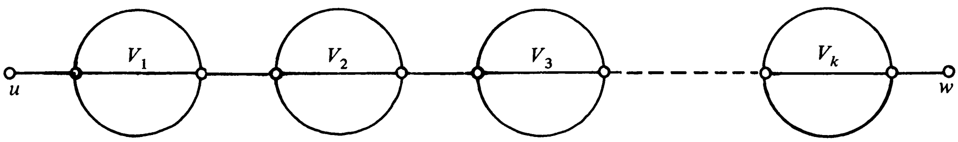

FIGURE 2

$x \in V$ and $S_V < \infty$. Let $t = c(x, v_0, v_1)$ (notation as at the end of §2). By **[Ca**, Proposition 1.2], $d(v_0, t) \geq d(v_1, t)$. Let now $y = X_{S_0 - 1}$, and let $q = c(y, v_0, v_1)$. Observe that q cannot lie on the geodesic from u to v_0 unless $q = v_0$. Therefore v_0 lies between u and y: the definition of x now implies that $d(v_0, q) < d(v_1, q)$. In other words, the vertices q and t lie in this order between v_0 and v_1. Therefore $s \geq d(y, x) = d(y, q) + d(q, t) + d(t, x)$ and $d(v_0, q) + d(q, t) + d(t, x) = d(v_0, x) \geq s$. Hence $d(y, q) \leq d(v_0, q)$, so that $d(y, y_0) \leq 2d(v_0, q) \leq s - 1$. Moreover, $d(y, v_1) = d(y, q) + d(q, v_1) \leq d(v_0, q) + d(q, v_1) = s - 1$. These inequalities say that $y = X_{S_0 - 1} \in V$, and therefore $S_V \leq S_0 < \infty$. □

The strong Markov property now yields

COROLLARY 1. *If V is an intermediate set lying between u and w, then*

$$F(u, w) = (Q_V F)(u, w) = \sum_{v \in V} \alpha^V(u, v) F(v, w) = A(\{u\}, V)\mathbf{f}(V, w).$$

Now let $V_1, \ldots, V_k$ be intermediate sets lying between u and w such that $d(V_{i+1}, u) \geq d(V_i, u) + (M+1)s$ (see Figure 2), and let $A_i = A(V_i, V_{i+1})$. Iterating the formula of Corollary 1, we obtain

$$(3.2) \quad F(u, w) = (Q_{V_1} Q_{V_2} \cdots Q_{V_k} F)(u, w) = A(\{u\}, V_1) A_1 A_2 \cdots A_{k-1} \mathbf{f}(V_k, w).$$

If w tends to an end of T, the length of the matrix product in (3.2) tends to infinity. We need to prove a *Perron-Frobenius* type convergence theorem for these sequences of matrix products. For this goal, we study the properties of intermediate matrices. By (3.1), the intermediate matrices of T have uniformly bounded finite dimensions.

LEMMA 5. *Let $A = A(V, W)$ be an intermediate matrix. Then, for v, v' in V and w in W:*

$$\alpha^W(v', w) \geq \delta^{M\, d(v, v')} \alpha^W(v, w).$$

PROOF. Observe that V contains all vertices of the geodesic which connects v and v'. Therefore we may assume for the proof, that v and v' are neighbors. By Lemma 3, for w in W, the function $h = Q_W \delta_w$ is harmonic outside $W' = \{w' \in T | d(w', W) \leq s\}$. In addition, by Definition 1, $d(V, W') > (M-1)s$. Therefore Lemma 2 implies $h(v') \geq \delta^M h(v)$. □

COROLLARY 2. *If $A = A(V, W)$ is an intermediate matrix, then the zeros of A are disposed in columns, that is, $\alpha^W(v_0, w) = 0$ for some $v_0 \in V$ implies $\alpha^W(v, w) = 0$ for all v in V.*

The intermediate matrices give rise to mappings between the positive (closed) cones of the finite-dimensional vector spaces where they act. We shall consider these mappings modulo projective equivalence. For $m \in \mathbf{N}$, let $\mathcal{C}_m = \{\mathbf{a} = (\alpha_1, \ldots, \alpha_m)^t \in \mathbf{R}^m | \alpha_i \geq 0\}$, $\mathcal{S}_m = \{\mathbf{a} \in \mathcal{C}_m | \sum_{i=1}^m \alpha_i = 1\}$, $\mathcal{C}_m^0 = \{\mathbf{a} \in \mathcal{C}_m | \alpha_i > 0$ for all $i\}$ and $\mathcal{S}_m^0 = \mathcal{S}_m \cap \mathcal{C}_m^0$. By $\mathcal{P}_m$ we denote the central projection from the origin of the nonzero elements of $\mathcal{C}_m$ onto $\mathcal{S}_m$. On $\mathcal{S}_m^0$ (or on the projective space $\mathcal{C}_m^0$) we consider the hyperbolic metric

$$\theta_m(\mathbf{a}, \mathbf{b}) = |\log\langle \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{c}'\rangle|, \qquad \mathbf{a}, \mathbf{b} \in \mathcal{S}_m^0,$$

where $\mathbf{c}$ and $\mathbf{c}'$ are the points where the line spanned by $\mathbf{a}$ and $\mathbf{b}$ intersects $\partial\mathcal{S}_m = \mathcal{S}_m - \mathcal{S}_m^0$, and $\langle \cdot, \cdot, \cdot, \cdot \rangle$ dentotes cross ratio (see [**BK**, **Bi**, **De**]). In other words, if $\mathbf{a} = t\mathbf{c} + (1-t)\mathbf{c}'$ and $\mathbf{b} = t'\mathbf{c} + (1-t')\mathbf{c}'$ $(0 < t, t' < 1)$, then

$$\theta_m(\mathbf{a}, \mathbf{b}) = \left| \log \frac{t}{1-t} - \log \frac{t'}{1-t'} \right|.$$

θ_m induces the Euclidean topology on $\mathcal{S}_m^0$.

We are now able to prove our version of the Perron-Frobenius theorem, which involves a uniform bound for the contraction ratios of intermediate matrices in the hyperbolic metric. This bound is given by the following lemma.

LEMMA 6. *Let $0 < \gamma < 1$ and $N > 0$. There exists $0 < \Delta < \infty$ such that the following holds: for $m, q \leq N$ and for every nonzero nonnegative $m \times q$ matrix A whose entries satisfy $\gamma a_{ij} \leq a_{kj} \leq \gamma^{-1} a_{ij}$, one has* $\operatorname{diam} \mathcal{P}_m(A\mathcal{S}_q - \{0\}) \leq \Delta$ *(hyperbolic diameter). Moreover, if $\mathbf{a}, \mathbf{b} \in \mathcal{S}_q^0$, then*

$$\theta_m(\mathcal{P}_m A\mathbf{a}, \mathcal{P}_m A\mathbf{b}) \leq \tanh(\Delta/4) \cdot \theta_q(\mathbf{a}, \mathbf{b}).$$

PROOF. Let $\mathcal{E}_m = \{\mathbf{x} \in \mathcal{S}_m | \gamma x_i \leq x_j \forall i, j \leq m\}$. The set $\mathcal{P}_m(A\mathcal{S}_q - \{0\})$ is spanned by the projection onto $\mathcal{S}_m$ of the nonzero column vectors of A, hence it is contained in the convex set $\mathcal{E}_m$. For the first part of the statement, let Δ_m be the hyperbolic diameter of $\mathcal{E}_m$ and choose $\Delta = \max\{\Delta_m | m \leq N\}$. The last inequality follows from Lemma 1 of [**Bi**, §4], applied to the segment $[\mathbf{a}, \mathbf{b}]$ and its image. □

THEOREM 1. *Let A_n be a sequence of $m_n \times m_{n+1}$ matrices satisfying the properties of Lemma 6. Then there is a vector $\mathbf{f}$ in $\mathcal{S}_{m_1}^0$ such that, for every sequence $\mathbf{a}_n$ in $\mathcal{S}_{m_n}^0$,*

$$\lim_{n \to \infty} \mathcal{P}_{m_1} A_1 A_2 \cdots A_{n-1} \mathbf{a}_n = \mathbf{f}.$$

PROOF. It is enough to observe that

$$\mathcal{P}_{m_1} A_1 \cdots A_n \mathcal{S}_{m_{n+1}}^0 = \mathcal{P}_{m_1} A_1 \cdots A_{n-1} \mathcal{P}_{m_n} A_n \mathcal{S}_{m_{n+1}}^0 \subseteq \mathcal{P}_{m_1} A_1 \cdots A_{n-1} \mathcal{E}_{m_n}.$$

The hyperbolic diameter of the latter set is not larger than $\Delta \cdot (\tanh(\Delta/4))^{n-1}$ by Lemma 6, and tends to zero as $n \to \infty$. □

In view of (3.1) and Lemma 5, Theorem 1 can be applied to intermediate matrices. Finally, we shall need the following result concerning the operators Q_V.

PROPOSITION 1. *Let V and W be two intermediate sets with $d(V, W) > Ms$. Let f be a nonnegative function on T. If $Q_V Q_W f = f$ on W and $Q_W Q_V f = f$ on V, then f vanishes identically either on V or on W.*

PROOF. Denote by $\mathbf{1}$ the constant function with value one on T. It is enough to show that either $Q_V Q_w \mathbf{1} < \mathbf{1}$ strictly on W or $Q_W Q_V \mathbf{1} < \mathbf{1}$ strictly on V. Indeed, if for example $Q_V Q_W \mathbf{1} < \mathbf{1}$ on W, then for some $0 < \varepsilon < 1$,

$$\max\{f(w)|w \in W\} = \max\{Q_V Q_W f(w)|w \in W\} \leq \varepsilon \max\{f(w)|w \in W\}.$$

By Corollary 2, the sets $W^* = \{w \in W|Q_W \delta_w(v_0) > 0\}$ and $V^* = \{v \in V|Q_V \delta_v(w_0) > 0\}$ do not depend on the particular choice of $v_0 \in V$ and $w_0 \in W$, respectively. Suppose there are $v_0 \in V$, $w_0 \in W$ such that $Q_W Q_V \mathbf{1}(v_0) = Q_V Q_W \mathbf{1}(w_0) = 1$. Then, since Q_W is a substochastic operator, $Q_{V^*}\mathbf{1}(w) = Q_V \mathbf{1}(w) = 1$ for every $w \in W^*$, and similarly, $Q_{W^*}\mathbf{1}(v) = Q_W \mathbf{1}(v) = 1$ for every $v \in V^*$. In other words, from every point in V^* the random walk visits with probability one some point in W^*, and vice versa. Thus, starting in V^*, an infinite number of returns to V^* takes place with probability one. This contradicts transience. □

Proposition 1 can also be stated as follows: among the two matrix products $A(V,W)A(W,V)$ and $A(W,V)A(V,W)$, at least one is strictly substochastic. Note that property (iii) is not necessary for the proof.

4. The Martin boundary. We are now ready to realize the Martin boundary of the random walk as the space Ω of all ends of T with the topology described in §2. First, we extend the Martin kernel to $T \times (T \cup \Omega)$.

The notation of the next proposition is as in §3.

PROPOSITION 2. *Let U be a finite subset of T, of cardinality $|U| = k$, and $\omega \in \Omega$. Then there exists a vector $\mathbf{f}(U,\omega)$ in S_k^0 such that, for every sequence (w_n) in T convergent to ω,*

$$\lim_{n\to\infty} \mathcal{P}_k \mathbf{f}(U, w_n) = \mathbf{f}(U,\omega).$$

Moreover, if (ω_n) is a sequence in Ω which converges to ω, then

$$\lim_{n\to\infty} \mathbf{f}(U,\omega_n) = \mathbf{f}(U,\omega).$$

PROOF. Let v be the junction of U and ω, that is, the first common vertex of all the geodesics from U to ω. Now choose a sequence (V_m) of intermediate sets lying between v and ω, with $d(V_{m+1}, v) \geq d(V_m, v) + (M+1)s$. Let $c_n = c(v, w_n, \omega)$ and let V_{m_n} be the last set in the above sequence which lies between v and c_n. The assumption $w_n \to \omega$ is equivalent to $d(c_n, v) \to \infty$. Therefore, $m_n \to \infty$ as $n \to \infty$. Now (3.2) yields, for every $j \leq m_n$,

$$\mathbf{f}(U, w_n) = A(U,V_1)A(V_1,V_2)\cdots A(V_{j-1},V_j)\mathbf{f}(V_j, w_n). \tag{4.1}$$

If we choose $j = m_n$, then by Theorem 1 the right side converges in direction to a vector which is independent of the choice of the sequence (w_n) in T convergent to ω.

To prove the second statement, fix j in (4.1) and let $n \to \infty$. Then

$$\mathbf{f}(U,\omega) = \mathcal{P}_k A(U,V_1)A(V_1,V_2)\cdots A(V_{j-1}V_j)\mathbf{f}(V_j,\omega). \tag{4.2}$$

Let $c'_n = c(v,\omega,\omega_n)$. As $\omega_n \to \omega$, one has $d(v, c'_n) \to \infty$. Therefore we may replace ω by ω_n in (4.2). Now, by the same argument as above, $\mathbf{f}(U,\omega_n) \to \mathbf{f}(U,\omega)$ as $n \to \infty$. □

COROLLARY 3. *The Martin kernel extends continuously to $T \times (T \cup \Omega)$. For every ω in Ω, the extended function $K(\cdot, \omega)$ is harmonic.*

PROOF. Choose u in T and let $U = \{e, u\}$. For ω in Ω, write $\mathbf{f}(U, \omega) = (f(e, \omega, f(u, \omega))^t$. Then, by Proposition 2, $K(u, w_n)$ converges to

$$K(u, \omega) \stackrel{\text{def}}{=} f(u, \omega)/f(e, \omega) \tag{4.3}$$

for every sequence (w_n) in T which converges to ω. Thus the Martin kernel extends to $T \times (T \cup \Omega)$. Its continuity follows from the second statement in Proposition 2. Finally, we prove that $K(\cdot, \omega)$ is harmonic. For all u, w in T, by (2.2) and (2.3),

$$K(u, w) = \sum_{v \in T} p(u, v) K(v, w) + \frac{\delta_u(w)}{G(e, w)} \tag{4.4}$$

where δ_u is the Dirac measure at u. The summation on the right is finite by (iii): the result now follows by letting $w \to \omega$. □

Lemma 4 and Proposition 2 yield

COROLLARY 4. *For u in T, x in $T \cup \Omega$ and for every intermediate set V between u and x, $K(u, x) = (Q_V K)(u, x)$.*

PROPOSITION 3. *If $x \neq y$ in $T \cup \Omega$, then $K(\cdot, x) \neq K(\cdot, y)$.*

PROOF. Suppose $x \neq y$ and $K(u, x) = K(u, y)$ for all u in T. If both $x, y \in T$ this implies, as in [**KSK**, **De**], $F(x, y)F(y, x) = 1$, in contradiction with the transience of the random walk. If $x \in T$ and $y \in \Omega$, then by (4.4), $K(\cdot, x)$ is not harmonic, whereas $K(\cdot, y)$ is harmonic (Corollary 3). Therefore we may assume that $x = \omega_1$ and $y = \omega_2$ are both in Ω. Suppose $K(\cdot, \omega_1) = K(\cdot, \omega_2) = h$. Let $c = c(e, \omega_1, \omega_2)$, and choose intermediate sets V_i lying between c and ω_i respectively ($i = 1, 2$), with $d(V_1, V_2) \geq Ms$. Then, by Corollary 4, $Q_{V_1} h = h$ on V_2 and $Q_{V_2} h = h$ on V_1. Therefore $Q_{V_1} Q_{V_2} h = h$ on V_2 and $Q_{V_2} Q_{V_1} h = h$ on V_1. This contradicts Proposition 1. □

COROLLARY 5. *If (x_l) is a sequence in $T \cup \Omega$ such that $K(u, x_l)$ converges for every u in T, then (x_l) converges in $T \cup \Omega$.*

PROOF. If $(x_{l'})$ is a subsequence of (x_l) converging to $x \in T \cup \Omega$, then $K(u, x_{l'})$ converges to $K(u, x)$ by Corollary 3. By Proposition 3, (x_l) cannot have another accumulation point apart from x. As $T \cup \Omega$ is compact, this implies the statement. □

In view of the construction of the Martin boundary, described in §2, and in particular of (2.7), now Corollaries 3, 5 and Proposition 3 yield our main result.

THEOREM 2. *The Martin boundary of the random walk is the space Ω of all ends of T, equipped with the topology described in* §2.

The following result (compare [**De**, Theorem 3]) shows that all points of the boundary are extreme.

PROPOSITION 4. *For each ω in Ω, $K(\cdot, \omega)$ is an extreme point in the convex set of positive harmonic functions with value* 1 *at* e.

PROOF. Let h_1, h_2 be positive harmonic functions with $h_i(e) = 1$, and $K(u, \omega) = t h_1(u) + (1 - t) h_2(u)$ for all u in T, $0 < t < 1$. For u in T and for any intermediate set V between u and ω, $h_i(u) \geq Q_V h_i(u)$, and, as in [**De**], Corollary 4 yields

$$h_i(u) = Q_V h_i(u). \tag{4.5}$$

We now return to the notation of Proposition 2 and its proof, with the following addition: for every finite set $W \subset T$, we denote by $\mathbf{h}_i(W)$ the vector $(h_i(w))_{w \in W}$, $i = 1, 2$. Then, for $U = \{e, u\}$ and $j > 0$, iterating (4.5) yields

$$\mathbf{h}_i(U) = A(U, V_1)A(V_1, V_2) \cdots A(V_{j-1}, V_j)\mathbf{h}_i(V_j). \tag{4.6}$$

Therefore, as in Proposition 2, the right hand side of (4.6) converges in direction to $\mathbf{f}(U, \omega)$: in other words, $h_i(u) = h_i(u)/h_i(e) = K(u, \omega)$ for $i = 1, 2$. The result now follows by [**KSK**, Lemma 10-30]. □

COROLLARY 6. *For every positive harmonic function h on T there is a unique Borel measure ν_h on Ω such that*

$$h(u) = \int_\Omega K(u, \omega)\nu_h(d\omega) \quad \textit{for all } u \textit{ in } T.$$

We conclude this section with a description of the boundary behavior of positive harmonic functions: we give a short proof, mimicking [**De**, Theorem 5], of a "nontangential Fatou theorem" for a random walk on T with properties (i), (ii) and (iii). We point out that an alternative proof of this theorem may be derived using Harnack's inequality and the theory of fine convergence, along the lines of [**KP**, §3]. This alternative proof does not make use of the existence of the boundary limit in (2.9). As in [**KP**], for every integer $\kappa \geq 0$ and for every ω in Ω, we denote by $\Gamma_\kappa(\omega)$ the *cone of width κ centered at ω*, that is, the set

$$\Gamma_\kappa(\omega) = \{u \in T | \exists \kappa \geq 0 \colon d(u, v_k) \leq \kappa\}, \tag{4.7}$$

where v_k, $k = 0, 1, 2, \ldots$, are the vertices of the geodesic from e to ω. By ν we denote the unique probability measure $\nu = \nu_1$ on Ω which represents the harmonic function with value 1.

THEOREM 3. *Given two positive harmonic functions g and h, let $\nu_h = \phi\nu_g + \sigma$ be the Lebesgue decomposition of ν_h with respect to ν_g: that is, $\phi \in L^1(\nu_g)$ and σ is singular with respect to ν_g. Then, for every $\kappa > 0$,*

$$\lim_u \frac{h(u)}{g(u)} = \phi(\omega)$$

for ν_g-almost every ω in Ω, if u tends to ω in $\Gamma_\kappa(\omega)$.

PROOF. We first reduce the proof to the case $g = \mathbf{1}$ (compare [**KSK**, Definition 10-23]). Since g is strictly positive, we may define new transition probabilities on T by $\tilde{p}(u, v) = p(u, v)g(v)/g(u)$. It is clear that these new transition probabilities satisfy (i) and (ii). Property (iii) holds with $\tilde{\delta} = \delta^{Ms+1}$ by Lemma 2. The Green function is now $\tilde{G}(u, v) = G(u, v)g(v)/g(u)$, and the Martin kernel is $\tilde{K}(u, v) = K(u, v)g(e)/g(u)$. Define $\tilde{h} = h/g$: then $\tilde{h}$ is harmonic with respect to $\tilde{p}$. It is easily seen that its representing measure (in the sense of Corollary 6, with respect to $\tilde{p}$) is $\nu_{\tilde{h}} = \nu_h/g(e)$. In particular, the representing measure of $\mathbf{1}$ is $\tilde{\nu} = \nu_g/g(e)$, and $\nu_{\tilde{h}} = \phi\tilde{\nu} + \sigma/g(e)$. Therefore it suffices to prove the theorem for the case $g = \mathbf{1}$.

Let λ be a trajectory of the random walk starting at e. Recall that $X_n(\lambda)$ denotes the random vertex at time n, and $X_\infty(\lambda) = \lim_n X_n(\lambda)$: the limit exists with probability one by (2.9). By the probabilistic Fatou theorem [**KSK**, Theorem 10-43],

$$\lim_{n\to\infty} h(X_n(\lambda)) = \phi(X_\infty(\lambda)) \quad \text{with probability one.} \tag{4.8}$$

On the other hand, let $\omega = X_\infty(\lambda)$, and denote by V_m, $m = 1, 2, \ldots$, the sequence of intermediate sets on the geodesic from e to ω with $d(e, V_1) = 1$ and $d(e, V_{m+1}) = d(e, V_m) + s$. Then, almost surely, the trajectory λ meets each V_m at least once. Given a vertex u in $\Gamma_\kappa(\omega)$, choose and fix the intermediate set among the V_m which is closest to u, and call it V_u. We observe that, for v in V_u, $d(v, u) \le \kappa + s = \kappa'$. Thus, by Harnack's inequality,

$$h(v) \ge \delta^{\kappa' M} h(u) \quad \text{for all } v \text{ in } V_u.$$

Now choose u_j in $\Gamma_\kappa(\omega)$, $d(e, u_j) \to \infty$, and define V_{u_j} as above. Denote by n_j the entrance time of the trajectory λ into the set V_{u_j}. Then $n_j \to \infty$ as $j \to \infty$, and

$$h(X_{n_j}(\lambda)) \ge C h(u_j) \tag{4.9}$$

with $C = \delta^{\kappa' M}$ independent of j.

Note that ν is the "hitting probability" of the random walk [**KSK**, Proposition 10-21]: if $B \subseteq \Omega$ is a Borel set, then $\nu(B) = \Pr_e[X_\infty \in B]$. Now the result follows by (4.8) and (4.9) in the case $\phi \equiv 0$, that is, when ν_h is singular with respect to ν.

It is now sufficient to handle the case $\nu_h \ll \nu$. If $B \subset \Omega$ is a Borel set and $d\nu_h/d\nu = \chi_B$, then (4.8) and (4.9) show that $\lim_j h(u_j) = 0$ if u_j converges to $\omega \in \Omega \setminus B$ in $\Gamma_\kappa(\omega)$ for ν-almost all ω. On the other hand, since $\nu_h \le \nu$, $h \le 1$ by Corollary 6, and $d\nu_{1-h}/d\nu = \chi_{\Omega - B}$. Therefore $\lim_j h(u_j) = 1$ if u_j converges to $\omega \in B$ in $\Gamma_\kappa(\omega)$ for ν-almost all ω. This proves the theorem when ϕ is a characteristic function of a Borel set. The remainder of the proof is the same as in [**De**, Theorem 5]: one extends the result to all functions ϕ which are uniform limits of simple functions and thus, by Egoroff's theorem, to all functions in $L^1(\nu)$. □

Positive eigenfunctions of the transition operator. The results of this section can be extended to all positive eigenfunctions of the transition operator. For $z \in \mathbf{C}$, functions h on T such that $zPh = h$ are called *z-harmonic.* For z in $\mathbf{C}$ and u, v in T, let

$$G(u, v|z) = \sum_{n=0}^{\infty} p^{(n)}(u, v) z^n. \tag{4.10}$$

It is known [**Sen**] that all power series $G(u, v|z)$, for u, v in T, have a common radius of convergence r, $1 \le r < \infty$, which is given by

$$r = 1 \Big/ \limsup_{n\to\infty} p^{(n)}(u, v)^{1/n}. \tag{4.11}$$

Moreover, convergence, or divergence respectively, of $G(u, v|z)$ at $z = r$ occur simultaneously for all u, v in T. In the first case, the random walk is called *r-transient*, in the second case we say X_n is *r-recurrent* (see, for instance, [**Sen**]).

By [**Pr**], positive z-harmonic functions exist if and only if $0 < z \le r$; in the r-recurrent case there is a unique positive r-harmonic function up to multiplication by constants. Again, a nonnegative z-harmonic function vanishes identically if it is not strictly positive. Now let $0 < z \lesssim r$, where r may be included in the r-transient case and is excluded otherwise, and choose a positive z-harmonic function h. Introduce a new transition kernel $\tilde{p}(u,v) = zp(u,v)h(v)/h(u)$. All arguments of this section apply to $\tilde{P}$, as in the first part of the proof of Theorem 3. Consider the generalized Martin kernel

$$K(u,v|z) = G(u,v|z)/G(e,v|z). \tag{4.12}$$

Then the following results hold.

COROLLARY 7. *Let* $0 < z \lesssim r$. (a) $K(u,v|z)$ *extends to a continuous function on* $T \times (T \cup \Omega)$.

(b) *For every* ω *in* Ω, $K(\cdot,\omega|z)$ *is a* z*-harmonic function and is extreme in the convex set of positive* z*-harmonic functions with value* 1 *at* e.

(c) *Let* h *be a positive* z*-harmonic function on* T, $0 < z \lesssim r$. *Then there is a unique Borel measure* ν_h *on* Ω *such that*

$$h(u) = \int_\Omega K(u,\omega|z)\nu_h(d\omega) \quad \textit{for all } u \textit{ in } T.$$

Thus all the cones of positive z-harmonic functions, $0 < z \lesssim r$, can be identified with the cone of positive Borel measures on Ω. Finally, Theorem 3 holds for every pair of positive z-harmonic functions ($0 < z \lesssim r$).

5. Application to random walks on groups. In this section we consider a finitely generated, infinite discrete group Γ with unit element e and a probability measure μ onΓ which has finite support S and is *irreducible*, that is, S generates Γ as a semigroup. The (right) *random walk* on Γ with law μ is the Markov chain X_n, $n = 0,1,2,\dots$, with state space Γ and one-step transition probabilities

$$p(u,v) = \mu(u^{-1}v), \qquad u,v \in \Gamma. \tag{5.1}$$

Thus $Pf(u) = \sum_{v\in\Gamma} f(uv)\mu(v) = f * \check{\mu}(u)$, where $\check{\mu}(v) = \mu(v^{-1})$. The n-step transition probabilities are given by

$$p^{(n)}(u,v) = \mu^{(n)}(u^{-1}v), \tag{5.2}$$

where $\mu^{(n)}$ denotes the nth convolution power of μ and $\mu^{(0)} = \delta_e$, the Dirac measure at e.

In order to apply the results of the preceding section to an appropriate class of groups, we make some comments on the geometry of discrete groups.

Given a finite, symmetric set A of generators of Γ, the *Cayley graph* $C(\Gamma,A)$ is the graph with vertex set Γ and (nonoriented) edges $[u,ua]$, $u \in \Gamma$, $a \in A$ ($a \neq e$). We denote by d_A the distance in $C(\Gamma,A)$: $d_A(v,w)$ is the length (number of edges) of the shortest path in $C(\Gamma,A)$ connecting v and w. Note that $d_A(uv,uw) = d_A(v,w)$ for all $u \in \Gamma$. If B is another finite symmetric set of generators, and $L(A,B)$ is the minimum among all numbers K such that $B \subseteq \bigcup\{A^k | k \leq K\}$, then

$$d_A \leq L(A,B)d_B, \tag{5.3}$$

that is, the metrics d_A and d_B are equivalent.

An *infinite path* in $C(\Gamma,A)$ is a sequence $\pi = (v_0,v_1,v_2,\dots)$ of successively contiguous vertices (elements of Γ) without repetitions. Two infinite paths π and π' are *equivalent* if for any finite subset V of Γ there exists a finite path lying entirely outside V which connects a vertex of π with a vertex of π'. An *end* of Γ is an equivalence class of infinite paths. Observe that the ends of Γ do not depend on the particular choice of the generating set A: if π and π' represent two different ends (equivalence classes) in $C(\Gamma, A)$ and if B is another finite symmetric set of generators, then π and π' also represent two different ends in $C(\Gamma, A \cup B)$. A general treatment of the theory of ends of a group can be found in [**St**, **Fr**].

DEFINITION 2. We say that Γ admits a *uniformly spanning tree*, if there is a tree T with vertex set Γ, such that the metric $d = d_T$ of T is equivalent with the metric d_A for any finite symmetric set A of generators of Γ.

Thus for each Cayley graph $C(\Gamma, A)$ as described above there is a constant $L(T, A)$ such that

$$L(T,A)^{-1} d_A \le d_T \le L(T,A) d_A. \tag{5.4}$$

REMARK. Consider a Cayley graph $C(\Gamma, B)$ (B finite symmetric) of Γ. If $d(u,v) = 1$, then $d_B(u,v) \le L = L(T,B)$, and $u^{-1}v$ is contained in $A = \bigcup\{B^n | 1 \le n \le L\}$. Therefore T is a subtree of $C(\Gamma, A)$, that is, every edge of T is an edge of $C(\Gamma, A)$.

There is a bijection between the ends of T (in the sense of §2) and the ends of Γ. Following the above remark, we choose A in the next lemma such that T is a subtree of $C(\Gamma, A)$.

LEMMA 7. *Each end of* Γ (*considered as an equivalence class in* $C(\Gamma, A)$) *has a common representative with exactly one end of* T.

PROOF. (a) If $\pi = (v_0, v_1, v_2, \ldots)$ is an infinite path in $C(\Gamma, A)$, then there must be a neighbor w_1 of $w_0 = e$ in T, which belongs to infinitely many of the T-geodesics from e to v_i, $i = 0, 1, 2, \ldots$. By induction, we obtain an infinite geodesic $\omega = [w_0, w_1, w_2, \ldots]$ of T such that each w_k lies on infinitely many of the T-geodesics from e to v_i. Thus π and ω are equivalent as infinite paths in $C(\Gamma, A)$, and each end of Γ is represented by at least one end of T.

(b) Suppose that $\omega = [v_0 = e, v_1, v_2, \ldots]$ and $\omega' = [v_0' = e, v_1', v_2', \ldots]$ are two different infinite geodesics of T representing the same end of Γ. Let $v_k = c(e, \omega, \omega')$ be the confluent of the two geodesics in T (as defined in §2). Choose $n > k$ so that $d_T(v_n, v_k) \ge (L+1)L^2$ and $d_T(v_n', v_k) \ge (L+1)L^2$, where $L = L(T, A)$. Then $d_A(v_n, v_k) \ge (L+1)L$ and $d_A(v_n', v_k) \ge (L+1)L$. Since ω and ω' are assumed to be equivalent infinite paths, there are elements $v_n = w_0, w_1, \ldots, w_j = v_n'$ such that $d_A(w_{i-1}, w_i) = 1$ and $d_A(w_i, v_k) \ge (L+1)L$. Thus $d_T(w_i, v_k) \ge L + 1$. Moreover, each point u on the T-geodesic from w_{i-1} to w_i must satisfy $d_T(u, w_i) \le L$. Hence we have found a finite path in T connecting v_n and v_n', whose elements u satisfy $d_T(u, v_k) \ge 1$. As T is a tree, this contradicts the fact that v_k must lie on this path. □

Denote by Ω the set of all ends of Γ, or of T respectively, and endow $\Gamma \cup \Omega$ with the topology of $T \cup \Omega$, as described in §2. If T' is another uniformly spanning tree of Γ, then, by assumption, T and T' have equivalent metrics, hence they give rise to the same topology on Ω. The topology on the space of ends of a group can also be described without reference to uniformly spanning trees and independently of generating sets [**St**]. Γ acts on Ω as a group of homeomorphisms.

Now let μ be an irreducible probability measure on Γ with finite support S, as described above.

LEMMA 8. *If* Γ *admits a uniformly spanning tree* T, *then the random walk with law* μ, *considered as a walk on* T, *satisfies properties* (i), (ii) *and* (iii).

PROOF. Let $C(\Gamma, A)$ be a Cayley graph of Γ which contains T as a subtree (this is possible by the remark preceding Lemma 7).

(i) As A is finite, by irreducibility, there is a positive integer M such that $A \subseteq \bigcup_{k=1}^{M} S^k$. If $d_T(u,v) = 1$ then also $d_A(u,v) = 1$, because T is a subtree. In other terms, $u^{-1}v \in A$. Hence $p^{(k)}(u,v) = \mu^{(k)}(u^{-1}v) > 0$ for some $k \leq M$.

(ii) As the metric d_A is left invariant, and as S is finite, $p(u,v) > 0$ implies $d_A(u,v) \leq m_0 = \max\{d_A(e,u) | u \in S\}$. Hence $d_T(u,v) \leq s$, where $s = m_0 L(T,A)$.

Property (iii) is obvious. □

In view of Lemma 8, *all results of the preceding chapters carry over to finitely supported, irreducible random walks on groups* Γ *admitting a uniformly spanning tree, and the Martin boundary is the space* Ω *of all ends of* Γ.

Denote now by $\|\mu\|$ the norm of μ as a convolution operator on $l^2(\Gamma)$. Note that $\|\mu\| \geq 1/r$, where r is the common radius of convergence of the power series $G(u,v|z)$, $u, v \in \Gamma$, $z \in \mathbf{C}$, as defined in (4.10). Furthermore, if μ is symmetric, then $\|\mu\| = 1/r$ [**K1**]. Whereas the description of the Martin boundary and the nontangential Fatou theorem rely only on the tree-like structure of the Cayley graph, the following result makes use of the group structure (see also [**De**, **Ser**]) and does not remain true in general, as we shall see in §6. As in Theorem 3, let $\Gamma_\kappa(\omega)$ be the "cone of width κ centered at ω", defined with respect to a uniformly spanning tree.

PROPOSITION 5. *Choose* $0 < z < 1/\|\mu\|$. *Let* T *be a uniformly spanning tree of* Γ, $\omega_0 \in \Omega$, *and let* (u_n) *be a sequence in* Γ *converging to* ω_0 *in* $\Gamma_\kappa(\omega_0)$.

(a) *If* U *is a neighborhood of* ω_0 *in* Ω, *then* $\lim_{n\to\infty} K(u_n, \omega|z) = 0$ *uniformly for* $\omega \in \Omega - U$.

(b) $\lim_{n\to\infty} K(u_n, \omega_0|z) = \infty$.

We omit the proof: by observing that our choice of z implies $(G(u,v|z))_{v\in\Gamma} \in l^2(\Gamma)$ for all u, the argument is as in the proof of the easier part of [**De**, Lemma 5].

COROLLARY 8. *If* $\|\mu\| < 1$, *then the harmonic measure* $\nu = \nu_1$ *on* Ω *which represents the harmonic function* $h \equiv 1$ *is continuous and supported on the whole of* Ω.

PROOF. Let $\omega_0 \in \Omega$ be represented by the geodesic $[e = v_0, v_1, v_2, \ldots]$ in T, and let U be any neighborhood of ω_0 in Ω. Then, by Proposition 5(a),

$$1 = \lim_{n\to\infty} \int_\Omega K(v_n, \omega)\nu(d\omega) = \lim_{n\to\infty} \int_U K(v_n, \omega)\nu(d\omega),$$

and $\nu(U) > 0$. By Proposition 5(b), we cannot have $\nu(\omega_0) > 0$. □

Thus, if $\|\mu\| < 1$, then the Martin boundary and the Poisson boundary of the random walk coincide. Therefore, by [**K2**], Corollary 8 applies when Γ is nonamenable (and admits a uniformly spanning tree). Further results which rely upon the group structure, as for example the "solution of the Dirichlet problem" and the cocycle identity

$$K(xy, \omega|z) = K(x, \omega|z) K(y, x^{-1}\omega|z)$$

can be proved as in [**De**, **Ser**].

Finally, we turn to the question of how to construct groups satisfying Definition 3. It is known [**Fr**, **St**] that a finitely generated infinite group has one, two or infinitely many ends. Groups with one end, for example $\mathbf{Z}^2$, certainly have no uniformly spanning tree (their Cayley graphs cannot be disconnected by removing a finite subset, whereas this is possible for groups having a uniformly spanning tree). There are groups with infinitely many ends which admit no such tree, for example, the free product $\mathbf{Z}^2 * \mathbf{Z}^2$. Every group with two ends has a finite normal subgroup such that the factor group is isomorphic with $\mathbf{Z}$ (the infinite cyclic group) or $\mathbf{Z}_2 * \mathbf{Z}_2$ (the infinite dihedral group); therefore every such group has a uniformly spanning tree.

In order to give a construction principle for groups admitting a uniformly spanning tree, we briefly describe the concepts of free products with amalgamation and HNN-extensions of groups. See [**LS**, §IV.2] for a detailed treatment.

(A) Given two groups Γ_1 and Γ_2 having a common subgroup H, their *free product with amalgamation* (or, briefly, amalgam) over H, $\Gamma = \Gamma_1 *_H \Gamma_2$, is the quotient of the free product $\Gamma_1 * \Gamma_2$ obtained by identifying the elements of H in Γ_1 and Γ_2. If X_i are sets of representatives of the cosets in Γ_i/H with $e \in X_i$, $i = 1, 2$, then each element $u \in \Gamma$ can be written uniquely as

$$u = x_{i_1} x_{i_2} \cdots x_{i_m} h, \quad \text{where } m \geq 0,\ i_j \in \{1, 2\},\ i_{j+1} \neq i_j, \quad x_{i_j} \in X_{i_j} - \{e\} \text{ and } h \in H \tag{5.5}$$

[**LS**, Theorem 2.6, p. 187].

(B) Given a group Γ_0, a subgroup H and an isomorphism ϕ of H onto another subgroup of Γ_0, the *HNN-extension* of Γ_0 over H and ϕ is the group $\Gamma = \langle \Gamma_0, t | ht = t\phi(h) \forall h \in H \rangle$. In other words, Γ is the quotient of the free product $\Gamma_0 * \mathbf{Z}$ ($\mathbf{Z}$ the infinite cyclic group with generator t) obtained by the quasi-commuting relations $ht = t\phi(h)$ for all $h \in H$. If X and Y are sets of representatives, both containing e, of the cosets in Γ_0/H and $\Gamma_0/\phi(H)$, respectively, then each $u \in \Gamma$ can be uniquely written as

$$u = u_1 \cdots u_m g_0, \quad \text{where } g_0 \in \Gamma_0,\ m \geq 0 \text{ and } u_j = x_j t^{k_j} \text{ or } u_j = y_j t^{-k_j} \text{ with } k_j > 0, \quad x_j \in X,\ y_j \in Y \text{ and } x_j, y_j \neq e \text{ for } j > 1 \tag{5.6}$$

[**LS**, Theorem 2.1, p. 181].

Note that, by [**St**, 5.A.9], each finitely generated group with infinitely many ends can be written as an amalgam or as an HNN-extension over a *finite* subgroup.

THEOREM 4. (A) *Let* Γ_1 *and* Γ_2 *be groups admitting uniformly spanning trees* T_1 *and* T_2*, respectively, and let* H *be a finite common subgroup. Then* $\Gamma = \Gamma_1 *_H \Gamma_2$ *also has a uniformly spanning tree.*

(B) *Let* Γ_0 *be a group admitting a uniformly spanning tree, and let* H *be a finite subgroup. Then any HNN-extension* Γ *of* Γ_0 *over* H *also has a uniformly spanning tree.*

(C) *If* Γ *contains a finite normal subgroup, such that the factor group admits a uniformly spanning tree, then* Γ *also has a uniformly spanning tree.*

PROOF. (A) By the remark before Lemma 7, we can choose finite symmetric sets of generators A_i of Γ_i, such that T_i is a subtree of $C(\Gamma_i, A_i)$, $i = 1, 2$. As H is finite, we can also assume $H \subseteq A_i$. There are constants $L_i = L(T_i, A_i)$ such that, for the distances d_i of T_i, $d_{A_i} \leq d_i \leq L_i d_{A_i}$, $i = 1, 2$. With notation as in (5.5), we now construct a tree T for Γ, such that T is a subtree of $C(\Gamma, A)$, $A = A_1 \cup A_2$: the set of vertices of T is Γ, and $[u, v]$ is an edge of T if and only if we can write $u = x_{i_1} \cdots x_{i_m} u_{i_{m+1}}$, $v = x_{i_1} \cdots x_{i_m} v_{i_{m+1}}$, where the x_{i_j} are as in (5.5), $i_{m+1} \neq i_m$ and $[u_{i_{m+1}}, v_{i_{m+1}}]$ is an edge of $T_{i_{m+1}}$. If $u = x_{i_1} \cdots x_{i_m}$ is as in (5.5), then the geodesics from e to u in T is obtained as follows: start with the geodesic from e to x_{i_1} in T_{i_1}. The part of the geodesic between $y_{j-1} = x_{i_1} \cdots x_{i_{j-1}}$ and y_j, $j < m$, is a copy of the geodesic from e to x_{i_j} in T_{i_j}. The final part from y_{m-1} to u is a copy of the geodesic from e to $x_{i_m} h$ in T_{i_m}. By assumption, T is a subtree of $C(\Gamma, A)$, so that $d_A \leq d$, where d is the metric of T. If $d_A(u, v) = 1$, u as in (5.5), then $v = ua$ for some $a \in A$.

Case 1. $a \in A_{i_m}, x_{i_m} ha = x'_{i_m} h' \in \Gamma_{i_m} - H$. Then the geodesic from e to u and from e to v coincide at least up to the point $x_{i_1} \cdots x_{i_{m-1}}$, and the remaining parts lie entirely within the same copy of T_{i_m}. Hence

$$d(u, v) = d_{i_m}(x_{i_m} h, x_{i_m} ha) \leq L_{i_m} d_{A_{i_m}}(x_{i_m} h, x_{i_m} ha) = L_{i_m}.$$

Case 2. $a \in A_{i_m}, x_{i_m} ha = h' \in H$. Then the geodesic in T from u to v decomposes into a part from u to $y_{m-1} = x_{i_1} \cdots x_{i_{m-1}}$ (lying within a copy of T_{i_m}) and a part from y_{m-1} to u (lying within a copy of $T_{i_{m-1}}$). We have

$$d_{A_{i_m}}(e, x_{i_m} h) \leq d_{A_{i_m}}(e, h') + d_{A_{i_m}}(x_{i_m} ha, x_{i_m} h) = 2,$$

hence

$$\begin{aligned} d(u, v) &= d_{i_m}(x_{i_m} h, e) + d_{i_{m-1}}\left(x_{i_{m-1}}, x_{i_{m-1}} h\right) \\ &\leq L_{i_m} d_{A_{i_m}}(x_{i_m} h, e) + L_{i_{m-1}} d_{A_{i_{m-1}}}\left(x_{i_{m-1}}, x_{i_{m-1}} h\right) \leq 2L_{i_m} + L_{i_{m-1}}. \end{aligned}$$

Case 3. $a \in A_{i_m}$, $ha = x_{i_{m+1}} h', x_{i_{m+1}} \in X_{i_{m+1}} - \{e\}$, $i_{m+1} \neq i_m$. The same argument as in Case 2 yields

$$d(u, v) = d_{i_m}(h, e) + d_{i_{m+1}}\left(e, x_{i_{m+1}} h'\right) \leq L_{i_m} + 2L_{i_{m+n}}.$$

Thus, if we set $L = L(T, A) = L_1 + L_2 + \max\{L_1, L_2\}$, then $d_A(u, v) = 1$ implies $d(u, v) \leq L$, and (A) is proved.

(B) Choose a finite symmetric set A_0 of generators of Γ_0 containing H and $\phi(H)$, such that the uniformly spanning tree T_0 of Γ_0 is a subtree of $C(\Gamma_0, A_0)$. Set $A = A_0 \cup \{t, t^{-1}\}$. Using the notations of (5.6), we construct a tree T for Γ which is a subtree of $C(\Gamma, A) : [u, v]$ is an edge of T if and only if we can write u, v in one of the following three ways (up to switch of u and v):

(1) $u = u_1 \cdots u_m g_0$, $v = u_1 \cdots u_m g'_0$, $[g_0, g'_0]$ an edge of T_0,

(2) $u = u_1 \cdots u_m x_{m+1} t^l$, $v = u_1 \cdots u_m x_{m+1} t^{l+1}$, $l \geq 0$,

(3) $u = u_1 \cdots u_m y_{m+1} t^{-l}$, $v = u_1 \cdots u_m y_{m+1} t^{-l-1}$, $l \geq 0$,

where $u_1, \ldots, u_m$ are as in (5.6), $x_{m+1} \in X - \{e\}$, $y_{m+1} \in Y - \{e\}$; if $m = 0$ we may also have $x_1 = e$, $y_1 = e$. The proof now continues as in (A), factoring through several cases. (C) is obvious. □

Summarizing, the following construction principle yields a large class of groups admitting uniformly spanning trees:

(I) Start with finite groups, $\mathbf{Z}$, or $\mathbf{Z}_2 * \mathbf{Z}_2$.

(II) New groups can be constructed in a finite number of steps from groups already obtained by (A) amalgamation over a finite subgroup, (B) HNN-extension over a finite subgroup, (C) finite extensions in the sense of Theorem 6(C). For all these groups, the Martin boundary of any finitely supported, irreducible, transient random walk coincides with the space of ends of the respective group.

6. Remarks and examples. This final section is devoted to examples. The first two show that the assumptions of uniform irreducibility and bounded step-length are necessary for our approach. A third example shows that the Poisson boundary may be properly contained in the Martin boundary, contrary to the group-invariant case considered in §5.

We start by showing that uniform irreducibility cannot be replaced by the usual notion of irreducibility (which says that for each $u, v \in T$ there exists $k = k_{u,v} \in \mathbf{N}$, such that $p^{(k)}(u, v) > 0$).

EXAMPLE 1. Let $T = \mathbf{N}$ with edges $[i, i+1]$, $i = 0, 1, 2, \ldots$. Choose $0 < p < 1/2$. The Martin boundary of the translation invariant random walk on $\mathbf{Z}$ given by $\mu = p\delta_1 + (1-p)\delta_{-1}$ has two points. Transporting this walk to $\mathbf{N}$ by the map $n \mapsto 2n$ for $n \geq 0$, $n \mapsto 1 - 2n$ for $n < 0$, gives a random walk satisfying (ii) and (iii) but not (i). □

The second example concerns a random walk on a tree with two ends, whose Martin boundary consists of one point, due to the fact that the step-length is not bounded.

EXAMPLE 2. Let $T = \mathbf{Z}$ with edges $[i, i+1]$, $i = 0, \pm 1, \pm 2, \ldots$. Define the transition probabilities

$$p(i, i+1) = p(-i, -i-1) = 1/2 \quad \text{for } i = 0, 1, 2, \ldots,$$
$$p(i+1, i) = p(-i-1, -i) = p(i, -i) = p(-i, i) = 1/4 \quad \text{for } i = 1, 2, \ldots,$$

$p(j, k) = 0$ in all other cases. Properties (i) and (iii) are satisfied, but (ii) fails.

As $p(m, n) = p(-m, -n)$, the map $n \mapsto |n|$ transports the given walk to the random walk on $\mathbf{N}$ with transition probabilities

$$\bar{p}(n, n-1) = \bar{p}(n, n) = 1/4, \quad \bar{p}(n, n+1) = 1/2 \text{ for } n > 0, \quad \bar{p}(0, 1) = 1.$$

It is well known that $\bar{p}$ is transient. As $G(0,0) = \overline{G}(0,0) < \infty$, the original walk is also transient. Its Martin boundary has only one point. Indeed, we can realize this walk on a new tree T', with only one end, so that properties (i) to (iii) hold: the vertex set of T' is $\mathbf{Z}$, and the edges are $[n-1, n]$ and $[n, -n]$ for $n > 0$. □

These examples emphasize the geometric significance of properties (i) and (ii). Consider a Markov chain with state space T (not yet necessarily a tree). The *graph* of the Markov chain has vertex set T and (for our purpose, nonoriented) edges $[u, v]$, when $p(u, v) > 0$ or $p(v, u) > 0$. Our results apply when this graph is "almost" a tree, that is, when we can equip T with a tree structure such that properties (i) to (iii) hold. In particular, (i) and (ii) imply the existence of a "uniformly spanning tree" (in the sense of §5) for this graph.

It is noteworthy that the treatment of group-invariant random walks given in [**De**] and [**Ser**] for free groups and Fuchsian groups respectively, uses in a crucial fashion an additional property: that the Green function is square-summable. This property holds for all nonamenable groups.

As a final example, we exhibit a random walk on a tree $\overline{T}$, such that (1) the Green function does not belong to $l^2(\overline{T})$, (2) the support of the harmonic measure

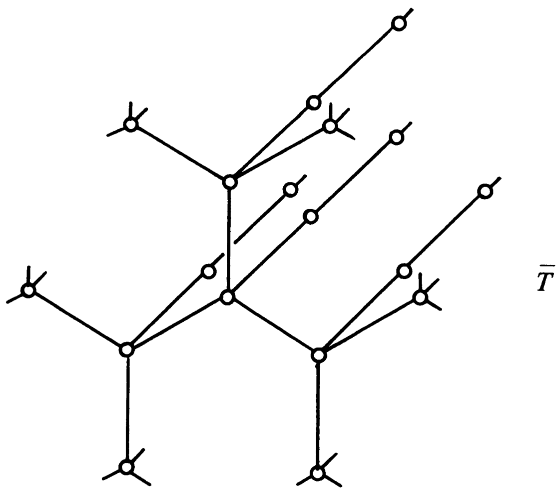

FIGURE 3

is smaller than the set of *all* ends and (3) $\overline{T}$ and the random walk are invariant under the nontransitive, but fixed point-free action of the nonamenable group $\mathbf{Z}_2 * \mathbf{Z}_2 * \mathbf{Z}_2$.

EXAMPLE 3. Let T be the homogeneous tree of degree 3 (with vertices u, v, w, etc.). At each vertex of T, we attach a half-line $\mathbf{N}$ to obtain a new tree $\overline{T}$ (see Figure 3). Observe that T is the Cayley-graph of $\mathbf{Z}_2 * \mathbf{Z}_2 * \mathbf{Z}_2$.

We describe the vertices of $\overline{T}$ by pairs uk, $u \in T$ and $k \in \mathbf{N}$, identifying $u0$ with u. Then the edges are of the types $[u, v] = [u0, v0]$ and $[uk, u(k+1)]$, where $[u, v]$ is an edge of T and $k \in \mathbf{N}$. We consider on $\overline{T}$ the *simple random walk* with transition probabilities $p(uk, vm) = 1/\text{degree}(uk)$, if uk and vm are contiguous, and $= 0$, otherwise. Properties (i), (ii), (iii) hold with $M = s = 1$ and $\delta = 1/4$, and the random walk is transient, since T is a transient subtree of $\overline{T}$. Note that $r = 1$ (compare [**GW**]). As the walk is of "nearest neighbor" type ($s = 1$), the results concerning the Martin boundary are already contained in [**Ca**]: in particular, the probabilities $F(uk, vm)$ are multiplicative along geodesics in $\overline{T}$. Using the methods of [**GW**], one can calculate the quantities $F(uk, vl)$ for neighbors uk, vl:

$$F(u0, v0) = \tfrac{1}{2} \quad \text{if } [u, v] \text{ is an edge in } T,$$

$$F(uk, u(k+1)) = \frac{k+1}{k+2}, \quad F(u(k+1), uk) = 1 \qquad \text{for } u \in T,\ k = 0, 1, 2, \dots .$$

We choose a reference vertex $e0 = e \in T$. The space $\overline{\Omega}$ of ends of $\overline{T}$ consists of the set Ω of ends of T and of the infinite geodesics $\omega_v = [e = u_0, \dots, u_j = v0, v1, v2, v3, \dots]$, where $v0 = v \in T$. Each point of $\Omega' = \overline{\Omega} - \Omega$ is isolated, and Ω' is a countable, open, dense subset of $\overline{\Omega}$. We describe the Martin kernel K on $\overline{T} \times \overline{\Omega}$: let $uk \in T$ and $\omega \in \overline{\Omega}$.

Case 1. $\omega \neq \omega_u$. Then the confluent $c(e, uk, \omega) = v0 = v$ is a point on the geodesic from e to u within T, and

$$K(uk, \omega) = K(u0, \omega) = (1/2)^{d(u,v) - d(e,v)}.$$

Case 2. $\omega = \omega_u, c(e, uk, \omega) = uk$ and

$$K(uk, \omega) = (1/2)^{-\operatorname{dist}(e,u)}(k+1).$$

In particular, for $\omega \neq \omega_u, \lim_{k\to\infty} K(uk, \omega) = K(u0, \omega) \neq 0$. Thus property (a) of Proposition 5 is not satisfied.

Further computations yield

$$F(uk, um) = 1 \quad \text{if } m \leq k, \quad \text{and} \quad = (k+1)/(m+1), \quad \text{if } m > k,$$
$$G(um, um) = \tfrac{8}{3} \quad \text{if } m = 0, \quad \text{and} \quad = 2(m+1), \quad \text{if } m > 0.$$

Hence, for fixed $uk \in \overline{T}$,

$$\sum_{vm\in\overline{T}} G(uk, vm)^2 \geq \sum_{m=k+1}^{\infty} F(uk, um)^2 G(um, um)^2 = \infty,$$

and, for fixed $vm \in \overline{T}$,

$$\sum_{uk\in\overline{T}} G(uk, vm)^2 \geq \sum_{k=m+1}^{\infty} F(vk, vm)^2 G(vm, vm)^2 = \infty.$$

Thus neither rows nor columns of the Green function are in $l^2(\overline{T})$. As $F(uk, um) = 1$ if $k \geq m$, the ends ω_u, $u \in T$, are what we would call "reflecting ends"; their hitting probabilities are zero, although each $\{\omega_u\}$ is open in $\overline{\Omega}$: $\Pr_e[X_\infty = \omega_u] = 0$. Indeed, as the simple random walk on $\mathbf{N}$ is recurrent, almost surely one can spend only a finite amount of time in any of the "hairs" $u\mathbf{N}$ in $\overline{T}$ sticking out of T.

In particular, one deduces that the support of the harmonic measure ν on $\overline{\Omega}$ which represents the harmonic function $h \equiv 1$ has support Ω (ν is the hitting probability, see [**KSK**]).

References

[**AC**] R. Azencott and P. Cartier, *Martin boundaries of random walks on locally compact groups*, Proc. 6th Berkeley Sympos. on Math. Statistics and Probability 3, Univ. of California Press, Berkeley, 1972, pp. 87–129.

[**Bi**] G. Birkhoff, *Extensions of Jentzsch's theorem*, Trans. Amer. Math. Soc. **85** (1957), 219–227.

[**BK**] H. Busemann and P. J. Kelly, *Projective geometry and projective metrics*, Academic Press, New York, 1953.

[**Ca**] P. Cartier, *Fonctions harmoniques sur un arbre*, Symposia Math. **9** (1972), 203–270.

[**De**] Y. Derriennic, *Marche aléatoire sur le groupe libre et frontière de Martin*, Z. Wahrsch. Verw. Gebiete **32** (1975), 261–276.

[**DM**] E. B. Dynkin and M. B. Malyutov, *Random walks on groups with a finite number of generators*, Soviet Math. Dokl. **2** (1961), 399–402.

[**Fr**] H. Freudenthal, *Über die Enden diskreter Räume and Gruppen*, Comment. Math. Helv. **17** (1944), 1–38.

[**Fu**] H. Furstenberg, *Random walks and discrete subgroups of Lie groups*, Advances in Probability and Related Topics, Vol. 1, Dekker, New York, 1971, pp. 1–63.

[**GW**] P. Gerl and W. Woess, *Simple random walks on trees*, European J. Combin. **7** (1986), 321–331.

[**KV**] V. A. Kaimanovich and A. M. Vershik, *Random walks on discrete groups: boundary and entropy*, Ann. Prob. **11** (1983), 457–490.

[**KSK**] J. G. Kemeny, J. L. Snell and A. W. Knapp, *Denumerable Markov chains*, 2nd ed., Springer-Verlag, New York, Heidelberg, Berlin, 1976.

[**K1**] H. Kesten, *Symmetric random walks on groups*, Trans. Amer. Math. Soc. **92** (1959), 336–354.

[**K2**] ——, *Full Banach mean values on countable groups*, Math. Scand. **7** (1959), 146–156.

[K3] ——, *The Martin boundary of recurrent random walks on countable groups*, Proc. 5th Berkeley Sympos. on Math. Statistics and Probability 2, Univ. of California Press, Berkeley, 1967, pp. 51–74.

[KP] A. Korányi and M. A. Picardello, *Boundary behavior of eigenfunctions of the Laplace operator on trees*, Ann. Scuola Norm. Sup. Pisa (to appear).

[LS] R. C. Lyndon and P. E. Schupp, *Combinatorial group theory*, Ergeb. Math. Grenzgeb. 89, Springer-Verlag, Berlin, Heidelberg, New York, 1977.

[Re] D. Revuz, *Markov chains*, North-Holland, Amsterdam, 1975.

[Sen] E. Seneta, *Non-negative matrices and Markov chains*, 2nd ed., Springer Series in Statistics, New York, Heidelberg, Berlin, 1981.

[Ser] C. Series, *Martin boundaries of random walks on Fuchsian groups*, Israel J. Math. **44** (1983), 221–242.

[St] J. Stallings, *Group theory and three-dimensional manifolds*, Yale Univ. Press, New Haven and London, 1971.

DIPARTIMENTO DI MATEMATICA "GUIDO CASTELNUOVO", UNIVERSITÀ DI ROMA, CITTÀ UNIVERSITARIA, I-00100 ROMA, ITALIA

INSTITUT FÜR MATHEMATIK UND ANGEWANDTE GEOMETRIE, MONTANUNIVERSITÄT, A-8700 LEOBEN, AUSTRIA (Current address of Wolfgang Woess)

Current address (M. A. Picardello): Dipartimento di Matematica, Universita dell'Aquila, 67100 L'Aquila, Italia